

The Calabi-Yau equation on the Kodaira-Thurston manifold¹

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Abstract

We prove that the Calabi-Yau equation can be solved on the Kodaira-Thurston manifold for all given T^2 -invariant volume forms. This provides support for Donaldson's conjecture that Yau's theorem has an extension to symplectic four-manifolds with compatible but non-integrable almost complex structures.

1 Introduction

A fundamental property of a compact Kähler manifold (M^n, ω) is that one can find Kähler metrics with prescribed volume form in a fixed Kähler class. This is known as Yau's Theorem [Y]. More precisely, given a Kähler class κ and a volume form σ with $\int_M \sigma = \kappa^n$, there exists a unique Kähler form $\tilde{\omega}$ in κ solving

$$\tilde{\omega}^n = \sigma. \quad (1.1)$$

We call (1.1) the Calabi-Yau equation.

In [D], Donaldson conjectured that Yau's theorem can be extended to the case of general symplectic four-manifolds with compatible almost complex structures, at least in the case $b^+ = 1$. Moreover, Donaldson outlined a program to use estimates for the Calabi-Yau equation and its generalizations to prove new results for four-manifolds. For a detailed discussion of this program and recent developments, we refer the reader to [D, W, TWY, LZ, TW, DLZ]. It was shown in [W] and [TWY] that many of Yau's estimates for (1.1) carry over to the non-Kähler setting. In particular, the Calabi-Yau equation can be solved if the Nijenhuis tensor of the almost complex structure is small in a certain sense [W] or if a curvature condition holds for the fixed metric [TWY]. In this paper we investigate (1.1) in the case of a well-known four-manifold: the Kodaira-Thurston manifold.

The Kodaira-Thurston manifold is given by $M = S^1 \times (\text{Nil}^3/\Gamma)$ where Nil^3 is the Heisenberg group

$$\text{Nil}^3 = \left\{ A \in GL(3, \mathbb{R}) \mid A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\},$$

and Γ is the subgroup of Nil^3 consisting of those elements of Nil^3 with integral entries, acting by left multiplication. Kodaira first investigated M in the 1950s, showing that

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M admits an integrable complex structure [K]. Thurston [Th] later observed that M also admits a symplectic form but no Kähler structure, the first manifold known to have this property. This manifold and its higher-dimensional generalizations have been thoroughly studied over the years, see for example [Ab, AD, CFG, FGG, M]. In this paper we will show that the Calabi-Yau equation can be solved on M assuming T^2 invariance. We make use of some ideas and estimates from [W] and [TWY].

Writing t for the S^1 coordinate, the 1-forms dx , dt , dy and $dz - xdy$ on $S^1 \times \text{Nil}^3$ are invariant under the action of Γ and thus define 1-forms on M . One can use these 1-forms to define a symplectic form

$$\Omega = dx \wedge dt + dy \wedge (dz - xdy)$$

on M and a compatible almost complex structure

$$J(dx) = dt, \quad J(dy) = dz - xdy.$$

The data (M, Ω, J) is thus an *almost-Kähler* manifold, but is not Kähler since J is not an integrable complex structure. Indeed the Kodaira-Thurston manifold cannot admit Kähler structures because $b^1(M) = 3$ [Th].

There is a T^2 -action on the Kodaira-Thurston manifold M which preserves Ω and J . Indeed, if we let S^1 and \mathbb{R} act on $S^1 \times \text{Nil}^3$ by translation in the t and z coordinates respectively then this action commutes with Γ and gives a free T^2 action on M preserving the 1-forms dx , dt , dy and $dz - xdy$. This is essentially the only free symplectic T^2 action on M [G].

Our main result is as follows.

Theorem 1.1 *Let σ be a smooth volume form on M , invariant under the T^2 -action given above and normalized so that $\int_M \sigma = \int_M \Omega^2$. Then there exists a unique T^2 -invariant symplectic form $\tilde{\omega}$ cohomologous to Ω , compatible with the almost complex structure J , solving the Calabi-Yau equation:*

$$\tilde{\omega}^2 = \sigma. \tag{1.2}$$

The uniqueness part of Theorem 1.1 is due to Donaldson [D] (see also [W]). Note that T^2 -invariance is not required for the uniqueness statement. On any 4-manifold equipped with an almost complex structure J , if $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are cohomologous symplectic forms compatible with J and satisfying $\tilde{\omega}_1^2 = \tilde{\omega}_2^2$ then $\tilde{\omega}_1 = \tilde{\omega}_2$.

In addition, one can recast Theorem 1.1 in terms of the Ricci form of the canonical connection. We explain this in Section 4 - see Theorem 4.1 below. This can be regarded as a kind of analogue of another formulation of Yau's theorem often referred to as the Calabi conjecture: any representative of the first Chern class of a Kähler manifold can be written as the Ricci curvature of a Kähler metric in a given Kähler class.

The outline of the rest of the paper is as follows. First, in Section 2, we prove the key *a priori* estimate for the Calabi-Yau equation on the Kodaira-Thurston manifold. Part of this argument involves a Moser iteration argument from [TWY]. We complete the proof of Theorem 1.1 in Section 3, using some estimates from [W, TWY] and also [D]. In Section 5 we end with some further remarks and questions.

2 An *a priori* estimate for the Calabi-Yau equation

In this section we derive a uniform *a priori* bound for a T^2 -invariant solution $\tilde{\omega}$ of the Calabi-Yau equation

$$\tilde{\omega}^2 = \sigma, \quad (2.1)$$

where σ is a fixed volume form and $\tilde{\omega}$ is compatible with J . We write $\sigma = e^F \Omega^2$ for a smooth T^2 -invariant function F so that the Calabi-Yau equation becomes

$$\tilde{\omega}^2 = e^F \Omega^2. \quad (2.2)$$

By the normalization of σ we have

$$\int_M e^F \Omega^2 = \int_M \Omega^2. \quad (2.3)$$

We now introduce some natural objects associated to the almost-Kähler manifold (M, Ω, J) . Write g for the almost-Kähler metric, given by

$$g(X, Y) = \Omega(X, JY),$$

where we let J act on vectors by duality with the following convention: if τ is a 1-form we let $\tau(JX) = -(J\tau)(X)$. We also let J act on 2-forms η by $(J\eta)(X, Y) = \eta(JX, JY)$.

We assume that the solution $\tilde{\omega}$ of the Calabi-Yau equation

$$\tilde{\omega}^2 = e^F \Omega^2$$

is cohomologous to Ω and so we can write

$$\tilde{\omega} = \Omega + da, \quad (2.4)$$

for a a 1-form. Since $\tilde{\omega}$ and Ω are T^2 -invariant, after averaging a by the T^2 -action, we may assume that a is also T^2 -invariant.

The solution $\tilde{\omega}$ of (2.1) is compatible with J and we write \tilde{g} for the associated almost-Kähler metric, given by

$$\tilde{g}(X, Y) = \tilde{\omega}(X, JY).$$

In this section we prove the following *a priori* bound on the metric \tilde{g} .

Theorem 2.1 *There is a uniform constant C depending only on $\inf_M \Delta F$ such that*

$$\mathrm{tr}_g \tilde{g} \leq C, \quad (2.5)$$

where Δ is the Laplace operator associated to g .

Proof A general T^2 -invariant 1-form a can be written

$$a = f_1 dx + f_2 dt + f_3 dy + f_4(dz - xdy),$$

where $f_i = f_i(x, y)$ (for $i = 1, \dots, 4$). Taking the exterior derivative:

$$\begin{aligned} da &= f_{2,x} dx \wedge dt + (f_{3,x} - f_{1,y} - f_4) dx \wedge dy - f_{2,y} dt \wedge dy \\ &\quad + f_{4,x} dx \wedge (dz - xdy) + f_{4,y} dy \wedge (dz - xdy), \end{aligned}$$

where here and henceforth letter subscripts denote partial derivatives. Compute

$$\begin{aligned} J(da) &= f_{2,x} dx \wedge dt + (f_{3,x} - f_{1,y} - f_4) dt \wedge (dz - xdy) \\ &\quad + f_{2,y} dx \wedge (dz - xdy) - f_{4,x} dt \wedge dy + f_{4,y} dy \wedge (dz - xdy). \end{aligned} \quad (2.6)$$

The condition that $\tilde{\omega}$ of the form (2.4) is compatible with J is equivalent to the equation

$$J(da) = da, \quad (2.7)$$

and by the above this reduces to the following system of differential equations:

$$f_{3,x} - f_{1,y} - f_4 = 0 \quad (2.8)$$

$$f_{2,y} - f_{4,x} = 0. \quad (2.9)$$

Thus we can write

$$\begin{aligned} \tilde{\omega} &= (1 + f_{2,x}) dx \wedge dt + (1 + f_{4,y}) dy \wedge (dz - xdy) \\ &\quad + f_{4,x} dx \wedge (dz - xdy) - f_{4,x} dt \wedge dy, \end{aligned} \quad (2.10)$$

and

$$\tilde{\omega}^2 = \{(1 + f_{2,x})(1 + f_{4,y}) - f_{4,x}^2\} \Omega^2, \quad (2.11)$$

and hence the Calabi-Yau equation (2.2) becomes

$$(1 + f_{2,x})(1 + f_{4,y}) - f_{4,x}^2 = e^F. \quad (2.12)$$

The basis of left-invariant vector fields dual to $\{dx, dt, dy, dz - xdy\}$ is $\{\partial_x, \partial_t, \partial_y + x\partial_z, \partial_z\}$. The matrix of g with respect to this basis is the identity, while the matrix of \tilde{g} is

$$\tilde{g} = \begin{pmatrix} 1 + f_{2,x} & 0 & f_{4,x} & 0 \\ 0 & 1 + f_{2,x} & 0 & f_{4,x} \\ f_{4,x} & 0 & 1 + f_{4,y} & 0 \\ 0 & f_{4,x} & 0 & 1 + f_{4,y} \end{pmatrix}. \quad (2.13)$$

The following lemma is the key ingredient of this paper and makes crucial use of the structure of the Kodaira-Thurston manifold.

Lemma 2.2 *Let $\tilde{\Delta}$ be the Laplace operator associated to \tilde{g} and define*

$$u = \frac{\text{tr}_g \tilde{g}}{2} = 2 + f_{2,x} + f_{4,y}. \quad (2.14)$$

Then

$$\tilde{\Delta}u \geq \inf_M \Delta F. \quad (2.15)$$

Proof of Lemma 2.2 A straightforward calculation shows that the Laplace operators of g and \tilde{g} respectively applied to a general T^2 -invariant function $\psi = \psi(x, y)$ are given by the formulae

$$\Delta\psi = \frac{2\Omega \wedge d(Jd\psi)}{\Omega^2} = \psi_{xx} + \psi_{yy} \quad (2.16)$$

and

$$\tilde{\Delta}\psi = \frac{2\tilde{\omega} \wedge d(Jd\psi)}{\tilde{\omega}^2} = \frac{1}{\nu} ((1 + f_{2,x})\psi_{yy} + (1 + f_{4,y})\psi_{xx} - 2f_{4,x}\psi_{xy}), \quad (2.17)$$

where $\nu = (1 + f_{2,x})(1 + f_{4,y}) - f_{4,x}^2 = e^F$.

Applying (2.16) and (2.17) to $\log \nu$ and u respectively, and making use of (2.9) we find

$$\tilde{\Delta}u = \Delta \log \nu + \frac{1}{\nu} \left(\frac{\nu_x^2}{\nu} + \frac{\nu_y^2}{\nu} + 2(-f_{2,xx}f_{2,yy} + f_{2,yx}^2 - f_{4,yy}f_{4,xx} + f_{4,yx}^2) \right). \quad (2.18)$$

Now

$$\nu = AB - D^2, \quad \nu_x = Af_{2,yy} + Bf_{2,xx} - 2Df_{2,yx}, \quad \nu_y = Af_{4,yy} + Bf_{4,xx} - 2Df_{4,xy}, \quad (2.19)$$

where $A = 1 + f_{2,x}$, $B = 1 + f_{4,y}$ and $D = f_{4,x}$. Thus

$$\nu_x = \text{tr}(\mathcal{L}), \quad \nu_y = \text{tr}(\mathcal{M}).$$

where

$$\mathcal{L} = \begin{pmatrix} A & -D \\ -D & B \end{pmatrix} \begin{pmatrix} f_{2,yy} & f_{2,xy} \\ f_{2,xy} & f_{2,xx} \end{pmatrix}, \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} A & -D \\ -D & B \end{pmatrix} \begin{pmatrix} f_{4,yy} & f_{4,xy} \\ f_{4,xy} & f_{4,xx} \end{pmatrix}.$$

On the other hand,

$$\nu(f_{2,xx}f_{2,yy} - f_{2,yx}^2) = \det(\mathcal{L}), \quad \nu(f_{4,xx}f_{4,yy} - f_{4,yx}^2) = \det(\mathcal{M}),$$

and so

$$\begin{aligned} & \nu_x^2 + \nu_y^2 + 2\nu(-f_{2,xx}f_{2,yy} + f_{2,yx}^2 - f_{4,yy}f_{4,xx} + f_{4,yx}^2) \\ &= (\text{tr}(\mathcal{L}))^2 - 2\det(\mathcal{L}) + (\text{tr}(\mathcal{M}))^2 - 2\det(\mathcal{M}) \geq 0. \end{aligned} \quad (2.20)$$

For the inequality of (2.20), we are using the following elementary fact from linear algebra. If P and Q are 2×2 symmetric matrices with P positive definite, then

$$(\operatorname{tr}(PQ))^2 - 2\det(PQ) \geq 0. \quad (2.21)$$

Indeed, a direct computation gives (2.21) in the case when P is the identity matrix. For general P , write $P = SS^T$. Then

$$(\operatorname{tr}(PQ))^2 - 2\det(PQ) = (\operatorname{tr}(S^TQS))^2 - 2\det(S^TQS) \geq 0, \quad (2.22)$$

since S^TQS is symmetric, thus establishing (2.21).

Combining (2.18) and (2.20) completes the proof of Lemma 2.2. Q.E.D.

We can now finish the proof of Theorem 2.1. Observe that

$$u = \frac{2\Omega \wedge \tilde{\omega}}{\Omega^2},$$

and thus

$$\int_M u\Omega^2 = 2 \int_M \Omega \wedge (\Omega + da) = 2 \int_M \Omega^2.$$

From this L^1 bound of u together with Lemma 2.2 we can apply the Moser iteration argument of [TWY, Theorem 1.4] to obtain $\|u\|_{C^0} \leq C$, where the constant C depends only on F . Although this argument is contained in [TWY], we include a brief sketch here for the reader's convenience. For $p > 0$, compute using the Calabi-Yau equation:

$$\begin{aligned} \int_M |\nabla_g u^{p/2}|^2 \Omega^2 &\leq C' \int_M u d(u^{p/2}) \wedge Jd(u^{p/2}) \wedge \tilde{\omega} \\ &= -\frac{C'p}{8} \int_M u^p (\tilde{\Delta}u) \tilde{\omega}^2 \\ &\leq C''p \int_M u^p \Omega^2, \end{aligned} \quad (2.23)$$

where we have used an integration by parts to go from the first to the second line, and Lemma 2.2 for the third line. Combining (2.23) with the Sobolev inequality applied to $u^{p/2}$ we obtain for any $p > 0$,

$$\|u\|_{L^{2p}} \leq C^{1/p} p^{1/p} \|u\|_{L^p},$$

and a straightforward iteration argument now gives

$$\|u\|_{C^0} \leq C \|u\|_{L^1} \leq C,$$

as required. Q.E.D.

3 Proof of the Main Theorem

We now complete the proof of the main theorem. Following the method proposed in [D, W] we solve the Calabi-Yau equation (2.1) using a continuity method.

For $t \in [0, 1]$, consider the family of equations

$$\tilde{\omega}_t^2 = e^{tF+c_t}\Omega^2, \quad \text{with } [\tilde{\omega}_t] = [\Omega], \quad (3.1)$$

where the symplectic form $\tilde{\omega}_t$ is compatible with J and c_t is the constant given by

$$\int_M e^{tF+c_t}\Omega^2 = \int_M \Omega^2. \quad (3.2)$$

We wish to show that (3.1) has a T^2 invariant solution for $t = 1$. Consider the set

$$\mathcal{T} = \{t \in [0, 1] \mid \text{there exists a smooth solution of (3.1) for } t' \in [0, t]\}.$$

Since $\tilde{\omega}_0 = \Omega$ solves (3.1) for $t = 0$ we see that $0 \in \mathcal{T}$. To prove the main theorem it suffices to show that \mathcal{T} is both open and closed in $[0, 1]$.

For the openness part, we first need a brief discussion on the cohomology of M . First observe that the Kodaira-Thurston manifold has $b^+(M) = 2$ (see, for example, equation (3.1) of [L1]). A basis for the space of g -harmonic self-dual 2-forms is given by Ω together with the symplectic form Ω_1 given by

$$\Omega_1 = dx \wedge (dz - xdy) + dt \wedge dy. \quad (3.3)$$

Notice that Ω_1 is T^2 -invariant, closed, of type $(2, 0) + (0, 2)$ and self-dual. In particular $J(\Omega_1) = -\Omega_1$.

In [LZ] a cohomology group $H_J^-(M)$ was introduced as the space of all cohomology classes in $H^2(M; \mathbb{R})$ that can be represented by closed forms of type $(2, 0) + (0, 2)$. This was further studied in [DLZ]. In our case $H_J^-(M)$ is 1-dimensional, generated by $[\Omega_1]$.

We also have that

$$\Omega_1 \wedge \Omega = 0, \quad \Omega_1^2 = \Omega^2,$$

and so Ω, Ω_1 span a maximal subspace $H^+ \subset H^2(M; \mathbb{R})$ on which the intersection form is positive definite. Proposition 1 of [D] (cf. [W]) shows that if there exists a solution of (3.1) for $t_0 \in [0, 1]$ then one can find a solution $\tilde{\omega}_t$ of the equation

$$\tilde{\omega}_t^2 = e^{tF+c_t}\Omega^2, \quad (3.4)$$

for t sufficiently close to t_0 , with $\tilde{\omega}_t$ lying in the subspace H^+ . We claim that $\tilde{\omega}_t$ lies in $[\Omega]$. Indeed, writing $\tilde{\omega}_t = \alpha_t\Omega + \beta_t\Omega_1 + da$ we see that

$$\int_M \Omega^2 = \int_M \tilde{\omega}_t^2 = \alpha_t^2 \int_M \Omega^2 + \beta_t^2 \int_M \Omega_1^2, \quad (3.5)$$

and

$$0 = \int_M \tilde{\omega}_t \wedge \Omega_1 = \beta_t \int_M \Omega_1^2, \quad (3.6)$$

giving $\alpha_t = 1$ and $\beta_t = 0$. For (3.6) we have used the fact that $\tilde{\omega}_t$ is of type $(1,1)$ and Ω_1 is of type $(2,0) + (0,2)$. Hence $\tilde{\omega}_t$ lies in $[\Omega]$, showing that the set \mathcal{T} is open. More generally the same argument applies to all 4-manifolds satisfying $\dim H_J^-(M) = b^+(M) - 1$ (see [LZ, DLZ, L2]). Moreover, since Ω and F have T^2 symmetry, the implicit function theorem argument of [D] shows that the solution $\tilde{\omega}_t$ for $t \in \mathcal{T}$ must have T^2 symmetry. To show that \mathcal{T} is closed it remains to prove that a solution $\tilde{\omega}_t$ of (3.1) is uniformly bounded in C^∞ , independent of t .

For convenience, we write F for $tF + c_t$ and $\tilde{\omega}$ for $\tilde{\omega}_t$. The symplectic form $\tilde{\omega}$ is of the form (2.4). Then the result of Theorem 2.1 shows that $\text{tr}_g \tilde{g}$ is bounded by a constant depending only on the $C^2(g)$ bound of F . We can now directly apply the argument of [W] or [TWY] to obtain a uniform Hölder bound on the solution $\tilde{\omega}$. The higher order estimates then follow from the argument given in [D] or [W] (see also [TWY]). This completes the proof of the main theorem.

4 The Ricci form of the canonical connection

We now show that the main theorem can be recast in terms of the Ricci form of a certain connection on M . In general, given any symplectic form ω compatible with J there is an associated *canonical connection* ∇ on M . This connection is uniquely determined by the properties that if g is the associated almost-Kähler metric then $\nabla g = 0 = \nabla J$ and the $(1,1)$ -part of the torsion of ∇ vanishes identically (see, for example, [TWY] and the references therein). The curvature form of this connection expressed with respect to a local unitary frame is a skew-Hermitian matrix of 2-forms $\{\Psi_j^i\}$, $(i, j = 1, 2)$. The 2-form

$$\text{Ric}(\omega, J) = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^2 \Psi_i^i,$$

is then closed and cohomologous to the first Chern class $c_1(M, J)$. We call this 2-form the Ricci form of the canonical connection.

Then the main theorem can be restated as follows² (cf. [TW, Conjecture 2.4] or Question 6.8 of [L2]).

Theorem 4.1 *Let F be a smooth T^2 -invariant function on M . Then there exists a T^2 -invariant symplectic form $\tilde{\omega}$ on M , compatible with the almost complex structure J , satisfying*

$$\text{Ric}(\tilde{\omega}, J) = -\frac{1}{2}d(JdF). \quad (4.1)$$

Proof We choose $\tilde{\omega}$ to be the solution of the Calabi-Yau equation (2.2) given by Theorem 1.1. Differentiating twice the logarithm of (2.2) gives

$$\text{Ric}(\tilde{\omega}, J) = -\frac{1}{2}d(JdF) + \text{Ric}(\Omega, J),$$

²The idea of reformulating the Calabi-Yau equation in terms of the Ricci form of the canonical connection was known to V. Apostolov and T. Drăghici shortly after the paper [TWY] appeared (see the discussion in [L2]).

as in [TWY, (3.16)] (and see also [GH, pag.72], [L2, Proposition 4.5]). It remains to show that $\text{Ric}(\Omega, J) = 0$. We choose the global left-invariant unitary coframe

$$\theta^1 = \frac{dx + \sqrt{-1}dt}{\sqrt{2}}, \quad \theta^2 = \frac{dy + \sqrt{-1}(dz - xdy)}{\sqrt{2}}.$$

We first claim that the connection 1-forms $\{\theta_j^i\}$ ($i, j = 1, 2$) of the canonical connection of g are given by

$$\theta_1^1 = 0, \quad \theta_2^1 = -\frac{\sqrt{-1}}{2\sqrt{2}}\theta^2, \quad \theta_1^2 = -\frac{\sqrt{-1}}{2\sqrt{2}}\overline{\theta^2}, \quad \theta_2^2 = \frac{\sqrt{-1}}{2\sqrt{2}}(\theta^1 + \overline{\theta^1}).$$

Indeed, the matrix $\{\theta_j^i\}$ is skew-Hermitian, and so defines a connection ∇ with $\nabla g = 0 = \nabla J$. The torsion 2-forms $\{\Theta^i\}$ ($i = 1, 2$) of ∇ are defined by the first structure equation

$$d\theta^i = -\theta_j^i \wedge \theta^j + \Theta^i.$$

Since we have

$$d\theta^1 = 0, \quad d\theta^2 = -\frac{\sqrt{-1}}{\sqrt{2}}dx \wedge dy = -\frac{\sqrt{-1}}{2\sqrt{2}}(\theta^1 \wedge \overline{\theta^2} - \theta^2 \wedge \overline{\theta^1} + \theta^1 \wedge \theta^2 + \overline{\theta^1} \wedge \overline{\theta^2}),$$

one readily sees that

$$\begin{aligned} \Theta^1 &= 0, \\ \Theta^2 &= -\frac{\sqrt{-1}}{2\sqrt{2}}\overline{\theta^1} \wedge \overline{\theta^2}, \end{aligned}$$

which have no $(1, 1)$ -part, proving the claim. We note here that since Ω is closed, the torsion $\{\Theta^i\}$ is equal to the Nijenhuis tensor of J [TWY].

The curvature $\{\Psi_j^i\}$ of ∇ is given by the second structure equation

$$d\theta_j^i = -\theta_k^i \wedge \theta_j^k + \Psi_j^i,$$

which gives

$$\begin{aligned} \Psi_1^1 &= -\frac{1}{8}\theta^2 \wedge \overline{\theta^2}, \\ \Psi_2^1 &= \frac{1}{8}\left(-\theta^1 \wedge \overline{\theta^2} + 2\theta^2 \wedge \overline{\theta^1} - 2\theta^1 \wedge \theta^2 - \overline{\theta^1} \wedge \overline{\theta^2}\right), \\ \Psi_1^2 &= \frac{1}{8}\left(-\theta^2 \wedge \overline{\theta^1} + 2\theta^1 \wedge \overline{\theta^2} + 2\overline{\theta^1} \wedge \overline{\theta^2} + \theta^1 \wedge \theta^2\right), \\ \Psi_2^2 &= \frac{1}{8}\theta^2 \wedge \overline{\theta^2}. \end{aligned}$$

Since $\Psi_1^1 + \Psi_2^2 = 0$, it follows that $\text{Ric}(\Omega, J) = 0$. Q.E.D.

We note here that the standard Ricci curvature of the Levi-Civita connection of g cannot be identically zero: if it were, the metric g would be almost-Kähler and Einstein with vanishing scalar curvature, and a result of Sekigawa [S] would imply that J is integrable.

5 Further remarks and questions

(1) In [TWY] it was shown that the inequality $\tilde{\Delta}u \geq -C$ holds assuming the nonnegativity of a certain tensor \mathcal{R} , which can be expressed in terms of the curvature of the canonical connection of the reference almost-Hermitian metric and the Nijenhuis tensor. However, we cannot directly apply this result to the case of the Kodaira-Thurston manifold since the tensor \mathcal{R} associated to (g, J) has negative components, as can be confirmed by a direct (and lengthy) computation.

(2) It would be interesting to know whether Theorem 1.1 holds for T^2 -invariant almost complex structures on the Kodaira-Thurston manifold other than J . Pushing this further, one could also investigate estimates for the Calabi-Yau equation in terms of a taming but non-compatible symplectic form. This could be used to address the conjecture of Donaldson that the existence of a taming symplectic form implies the existence of a compatible symplectic form (see [D] and also [TWY], [LZ], [DLZ]). We note that it is of course not sensible to ask this question for our given almost complex structure J , since Ω is already a compatible symplectic form.

(3) It would be desirable to remove the assumption of T^2 invariance in the statement of Theorem 1.1. However, the inequality of Lemma 2.2 does not seem to hold by a similar argument in this more general case and so other techniques may be needed.

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